

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 2	Daily Tutorial Sheet - 7
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91.(C) We have: $\frac{1}{\cos(\theta + \alpha)} + \frac{1}{\cos(\theta - \alpha)} = \frac{2}{\cos \theta} \Rightarrow \frac{\cos(\theta - \alpha) + \cos(\theta + \alpha)}{\cos(\theta + \alpha) \cdot \cos(\theta - \alpha)} = \frac{2}{\cos \theta}$

$$\Rightarrow (2\cos \theta \cdot \cos \alpha) \cdot \cos \theta = 2(\cos^2 \theta - \sin^2 \alpha) \Rightarrow \cos^2 \theta \cdot \cos \alpha = \cos^2 \theta - \sin^2 \alpha$$

$$\Rightarrow \cos^2 \theta = \frac{\sin^2 \alpha}{1 - \cos \alpha} = \frac{4 \sin^2 \frac{\alpha}{2} \cdot \cos^2 \frac{\alpha}{2}}{2 \sin^2 \frac{\alpha}{2}} = 2 \cos^2 \frac{\alpha}{2} \Rightarrow \cos \theta = \pm \sqrt{2} \cos \frac{\alpha}{2}$$

92.(B) $y = \cos \theta \left(\sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right) \Rightarrow y \sec \theta - \sin \theta = \sqrt{\sin^2 \theta + \sin^2 \alpha}$

Squaring to get: $y^2 \sec^2 \theta - 2y \tan \theta - \sin^2 \alpha = 0 \Rightarrow y^2 \tan^2 \theta - 2y \tan \theta + (y^2 - \sin^2 \alpha) = 0$

As $\tan \theta \in \mathbb{R}$, $\Rightarrow 4y^2 - 4y^2(y^2 - \sin^2 \alpha) \geq 0 \Rightarrow y^2 \leq 1 + \sin^2 \alpha \Rightarrow -\sqrt{1 + \sin^2 \alpha} \leq y \leq \sqrt{1 + \sin^2 \alpha}$

93. Let $x = \tan A$, $y = \tan B$, $z = \tan C$

then $x + y + z = xyz$ becomes :

$$\Rightarrow \tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C = 0$$

$$\Rightarrow \frac{\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C}{1 - \tan A \cdot \tan B - \tan B \cdot \tan C - \tan C \cdot \tan A} = 0 \Rightarrow \tan(A + B + C) = 0 \Rightarrow A + B + C = 2\pi$$

Now, $A + B + C = n\pi \Rightarrow 2A + 2B + 2C = 2n\pi$

Taking $\tan$ on both sides, we get: $\Rightarrow \frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \cdot \tan 2B \cdot \tan 2C}{1 - \tan 2A \cdot \tan 2C - \tan 2B \cdot \tan 2C - \tan 2C \cdot \tan 2A} = 0$

$$\Rightarrow \tan 2A + \tan 2B + \tan 2C - \tan 2A \cdot \tan 2B \cdot \tan 2C = 0$$

$$\Rightarrow \tan 2A + \tan 2B + \tan 2C = \tan 2A \cdot \tan 2B \cdot \tan 2C$$

$$\Rightarrow \frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C} = \frac{8 \tan A \cdot \tan B \cdot \tan C}{(1 - \tan^2 A)(1 - \tan^2 B)(1 - \tan^2 C)}$$

$$\Rightarrow \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

94. Let $x = \tan A$, $y = \tan B$, $z = \tan C$

as $xy + yz + zx = 1$ (given)

$$\Rightarrow \tan A \tan B + \tan B \tan C + \tan C \tan A = 1 \Rightarrow \tan C (\tan A + \tan B) = 1 - \tan A \tan B$$

$$\Rightarrow \frac{(\tan A + \tan B)}{1 - \tan A \tan B} = \frac{1}{\tan C} = \cot C = \tan\left(\frac{\pi}{2} - C\right) \Rightarrow \tan(A + B) = \tan\left(\frac{\pi}{2} - C\right) \Rightarrow A + B + C = \frac{\pi}{2}$$

LHS : $\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$

$$\frac{1}{2} \left[\frac{2 \tan A}{1 + \tan^2 A} + \frac{2 \tan B}{1 + \tan^2 B} + \frac{2 \tan C}{1 + \tan^2 C} \right] = \frac{1}{2} \left[\sin 2A + \sin 2B + \sin 2C \right]$$

$$\begin{aligned}
 &= \frac{1}{2} [2 \sin(A+B) \cos(A-B) + \sin 2C] = [2 \cos C \cos(A-B) + 2 \sin C \cos C] \\
 &= \frac{1}{2} [2 \cos C (\cos(A-B) + \sin C)] = \cos C \left(\cos(A-B) + \sin \left(\frac{\pi}{2} - (A+B) \right) \right) \\
 &= \cos C (\cos(A-B) + \cos(A+B)) = 2 \cos C \cos A \cos B
 \end{aligned}$$

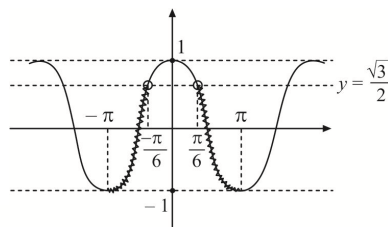
$$\begin{aligned}
 \text{RHS : } \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}} &= \frac{2}{\sqrt{(1+\tan^2 A)(1+\tan^2 B)(1+\tan^2 C)}} \\
 &= \frac{2}{\sec A \sec B \sec C} = 2 \cos A \cos B \cos C
 \end{aligned}$$

Hence proved

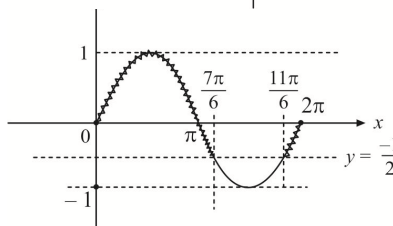
95.(B) $y + \cos x = \sin x \Rightarrow y = \sin x - \cos x = \sqrt{2} \sin \left(x - \frac{\pi}{4} \right)$

As $-1 \leq \sin \left(x - \frac{\pi}{4} \right) \leq 1 \Rightarrow -1 \leq \frac{y}{\sqrt{2}} \leq 1 \Rightarrow -\sqrt{2} \leq y \leq \sqrt{2}$

96.(A) $\cos x < \frac{\sqrt{3}}{2} \forall x \in [-\pi, \pi]$
 $\Rightarrow x \in \left[-\pi, -\frac{\pi}{6} \right) \cup \left(\frac{\pi}{6}, \pi \right]$



97.(B) $\sin x \geq \frac{-1}{2} \forall x \in [0, 2\pi]$
 $\Rightarrow x \in \left[0, \frac{7\pi}{6} \right] \cup \left[\frac{11\pi}{6}, 2\pi \right]$



98.(D) Consider numerator of the given expression

$$\begin{aligned}
 \text{i.e. } &\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10 \Rightarrow \cos 6\theta + 6 \cos 4\theta + 5 \cos 2\theta + 10 \cos 2\theta + 10 \\
 \Rightarrow &\cos 6\theta + \cos 4\theta + 5 \cos 4\theta + 5 \cos 2\theta + 10 \cos 2\theta + 10 \\
 \Rightarrow &(\cos 6\theta + \cos 4\theta) + 5(\cos 4\theta + \cos 2\theta) + 10(1 + \cos 2\theta) \\
 \Rightarrow &2 \cos 5\theta \cdot \cos \theta + 5 \cdot 2 \cdot \cos 3\theta \cdot \cos \theta + 10 \cdot 2 \cdot \cos^2 \theta \\
 \Rightarrow &2 \cos \theta [\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta] \quad \dots(i)
 \end{aligned}$$

By using (i) the given expression becomes, $\frac{2 \cos \theta [\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta]}{\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta} = 2 \cos \theta$

99.(C) Put $e^{\sin x} = t$

$\Rightarrow t = 2 \pm \sqrt{5}$

Either $\sin x = \log_e (2 + \sqrt{5})$

as $\log_e (2 + \sqrt{5}) > 1$

$\Rightarrow \sin x > 1$

Given equation becomes $t^2 - 4t - 1 = 0$

$\Rightarrow e^{\sin x} = 2 \pm \sqrt{5}$

or $\sin x = \log_e (2 - \sqrt{5})$

as $(2 - \sqrt{5})$ is (-ve) and log is not defined for (-ve) values

or no solution.

100.(AC) Let $81^{\sin^2 x} = y$..(i) and then $81^{\cos^2 x} = 81^{(1 - \sin^2 x)} = 81 \cdot 81^{-\sin^2 x} = 81 \cdot y^{-1}$...(ii)

So the given can be written as $y^2 - 30y + 81 = 0 \Rightarrow y = 3$ or $y = 27$

By using (i) and (ii),

$$\Rightarrow \text{Either } 81^{\sin^2 x} = 3 \quad \text{or} \quad 81^{\sin^2 x} = 27$$

$$\Rightarrow 4 \sin^2 x = 1 \quad \text{or} \quad 4 \sin^2 x = 3$$

$$\Rightarrow \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{\pi}{3} \quad \text{are only the solutions.}$$